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A CLASS OF GENERALIZED METZLERIAN MATRICES*

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1. This paper returns to a problem concerning the relationship between dynamic stability and Hicksian stability raised in a paper by Lloyd Metzler over twenty-five years ago [10]. The present paper identifies a class of matrices which has the property that dynamic stability implies Hicksian stability, as in the gross substitute or "Metzlerian" case. Further, as in the Metzlerian case, such matrices are specified in terms of their qualitative properties, i.e., their sign pattern configurations. Some links between this class of matrices and Samuelson's correspondence principle are also indicated.

2. It might be in order to give a brief review of work that has been done on Metzlerian matrices and variants of this class of matrices. In Value and Capital, Hicks' treatment of the competitive economy centered attention on two special cases: (a) the case where all commodities are gross substitutes, and (b) the case where all commodities obey the rules "substitutes of substitutes and complements of complements are substitutes while substitutes of complements and complements of substitutes

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are complements," all commodities being assumed to obey the law of demand in both cases. Hicks' analysis showed that, in small dimension cases at least, the assumption of Hicksian perfect stability (all i-th order principal minors have sign $(-1)^i$) implied certain comparative statics properties for the competitive economy. In particular, he asserted the famous Hicksian laws for the gross substitute economy, i.e., that an increase in demand for good i at expense of the numéraire (i) increased all equilibrium prices; (ii) increased the price of good i proportionately more than other prices. The proof of the proposition for many goods relied on the properties of composite commodities. Mosak [13] then presented a formal proof for the case of many goods using matrix analysis. Following this, Samuelson [16] raised the issue of the relationship between Hicksian and dynamic stability in the general case. Metzler's paper [10] was addressed to Samuelson's question, and contains a number of important propositions concerning dynamic stability and Hicksian stability. First, in a concise and brilliant argument, Metzler established that in the gross substitute case, Hicksian and dynamic stability are equivalent. Secondly, by use of counterexamples, the Metzler paper showed that in the general case, Hicksian stability is neither necessary nor sufficient for dynamic stability. Finally, Metzler proved that in what is now known as the case of "total stability," i.e., dynamic stability of any isolated subset of markets under any positive speeds of adjustment of prices, Hicksian stability characterizes the system of adjustment equations.

The next important breakthrough concerning the link between dynamic stability and Hicksian stability occurs in Morishima [11], where it was shown that Hicks' case (b) also has the property that dynamic stability is equivalent to Hicksian stability, and in addition Morishima derived a more general set of Hicksian laws of comparative statics to cover the presence of complements in the economy. One of the intriguing aspects of the work by Metzler and Morishima is that these results were obtained by economists unfamiliar with the Perron-Frobenius theorem,

and in fact represented extensions of this classical mathematical tool. The paper by Debreu and Herstein [5] summarized earlier findings concerning the Frobenius problem, and indicated generalizations and applications of particular interest to economists.

In the mid-1950s, Arrow and McManus [3] studied variants of the total stability problem posed by Metzler, with particular emphasis on the problem of D-stability, i.e., stability under all positive speeds of adjustment of markets. It might be mentioned that despite extended work on the problem of invariance of stability, the derivation of equivalent conditions for total stability and D-stability remains an unsolved problem.

Finally, Arrow and Hurwicz [2] and Arrow, Block and Hurwicz [1] in their treatment of stability of the competitive equilibrium introduced the properties of excess demand functions (homogeneity of degree zero in prices, Walras' Law, continuity and non-satiation) directly into the analysis of stability, obtaining as a major result the proof that under the gross substitutability assumptions, the competitive equilibrium is globally stable. McKenzie [9] provided an alternative proof of global stability in the gross substitute case, introducing the concept of a dominant diagonal matrix, a tool of central importance in a number of economic models. Morishima [12] extended the local Hicksian laws of comparative statics in the gross substitute case to global laws, under conditions guaranteeing global stability of the competitive economy.

3. The central concepts of this paper are the following. Given a real matrix A of dimension n x n, we say that A is Hicksian (or Hicksian stable) if every i-th order principal minor of A has sign $(-1)^i$ $i = 1, \cdots, n$. A is said to be a stable matrix if every characteristic root of A has real part negative. The basic question we are concerned with is that of identifying classes of matrices such that if a member of the class is stable then the matrix is Hicksian as well. Beyond the interest in this

question simply as an issue in the history of economic thought, there is the fact that while stability of A is the relevant property while analyzing convergence of an economic model, Hicksian stability is considerably more useful from the point of view of comparative statics analysis, and links between the results to be derived here and Samuelson's "correspondence principle" are also noted below.

Among the many results relating to Hicksian and dynamic stability, the following may be noted.[1]

(1) If A is a symmetric matrix, then A is stable if and only if A is Hicksian [12].

(2) If A is quasi-negative definite, then A is stable and A is Hicksian [12].

(3) If A is totally stable, then A is Hicksian [10].

(4) If A is Hicksian, then there exists a diagonal matrix D with diagonal elements positive such that DA is stable [6].

(5) If A is D-stable, then A is "almost" Hicksian, i.e., every i-th order principal minor of A has sign $(-1)^i$ or 0, with at least one principal minor of every order non-zero [15].

(6) In general, A Hicksian does not imply A is stable nor does A stable imply that A is Hicksian [10].

Historically, of particular interest to economists has been the analysis of stability and Hicksian stability in qualitatively specified matrices. The most important of these are Metzlerian matrices and Morishima matrices. For purposes of this paper, we define a Metzlerian matrix as a matrix A such that $a_{ii} < 0 \quad i = 1, \cdots, n$ and $a_{ij} \geq 0 \quad i \neq j, \; i, j = 1, \cdots, n$. A Morishima matrix, A, is a square matrix which can be permuted into the form

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$ where A_{11} and A_{22} are Metzlerian matrices

and $A_{12} \leq 0$, $A_{21} \leq 0$.

To further characterize a Morishima matrix, we use the concept of a cycle in a matrix (see [7, 8]). By a cycle in A (of length r) we mean a product of elements of A of the form $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r} a_{i_r i_1}$ where all indices $i_1, \cdots, i_r$ are distinct. As a matter of convention, diagonal elements in A are regarded as cycles in A of length one. We also define a chain in A (of length r-1) as a product of elements of A of the form $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r}$, where all indices $i_1, \cdots, i_r$ are distinct. We use the notation $a(i_1 \rightarrow i_r)$ to denote a chain from i_1 to i_r so that a cycle containing the indices i_1, i_r can be written as $a(i_r \rightarrow i_1) a(i_1 \rightarrow i_r)$. The importance of cycles in the analysis of stability (and Hicksian stability) of A stems from the determinantal formula developed by Maybee [7] which establishes that negative cycles of length r enter into the principal minor of order r with sign $(-1)^r$ while positive cycles of length r enter into the principal minor with sign $(-1)^{r+1}$.[2]

A link between cyclic analysis and Morishima matrices is the following. For A indecomposable, A is a Morishima matrix if and only if A satisfies (i) $a_{ii} < 0 \quad i = 1, \cdots, n$ and (ii) every cycle in A of length greater than one is non-negative (see [4]).[3] Clearly a Morishima matrix is a generalized version of a Metzlerian matrix in the sense that every Metzlerian matrix is also a Morishima matrix. Then the following results hold.

(7) If A is a Metzlerian matrix, then A is stable if and only if A is Hicksian [10].

(8) If A is a Morishima matrix, then A is stable if and only if A is Hicksian [11, 4].

In this paper we introduce a further generalization of Metzlerian (and Morishima) matrices that seems particularly appropriate for qualitatively specified economic models. We define a **generalized Metzlerian** (GM) matrix as follows.

Let $I = (i_1, \cdots, i_r)$ denote the index set of a negative cycle in an $n \times n$ real matrix A and let $J = (j_1, \cdots, j_s)$ denote the index set of a positive cycle in A. Then A is said to be a generalized Metzlerian (GM) matrix if (i) $a_{ii} < 0$ $i = 1, \cdots, n$, and (ii) given any I and J as defined, either $I \cap J = \emptyset$ or $I \subseteq J$.

The concept of a GM matrix can be illustrated by the following examples.

$$1 \quad \begin{bmatrix} - & + & + & + & + \\ + & - & + & + & + \\ + & + & - & + & + \\ + & + & + & - & + \\ + & + & + & + & - \end{bmatrix} \qquad 2 \quad \begin{bmatrix} - & + & - & - & - \\ + & - & - & - & - \\ - & - & - & + & + \\ - & - & + & - & + \\ - & - & + & + & - \end{bmatrix}$$

$$3 \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & - & - \\ 0 & 0 & 0 & + & - \end{bmatrix} \qquad 4 \quad \begin{bmatrix} - & - & + & + & + \\ + & - & 0 & 0 & 0 \\ 0 & + & - & + & + \\ 0 & + & + & - & + \\ 0 & + & + & + & - \end{bmatrix}$$

$$5 \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ 0 & - & + & 0 & 0 \\ 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & + \\ - & 0 & 0 & 0 & - \end{bmatrix} \qquad 6 \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & 0 & 0 \\ 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & - & - \\ + & 0 & - & 0 & - \end{bmatrix}$$

Example 1 is a Metzlerian matrix, while example 2 is a Morishima matrix. Since neither matrix contains any negative cycles of length greater than one, the GM conditions follow immediately. Example 3 is a "sign stable" matrix, i.e., one such that any matrix of this sign pattern is stable (see [15]). Such a matrix contains only negative cycles, hence is a member of the GM class. Example 4 is a non-trivial extension of the Metzlerian matrix, since $a_{12}a_{21}$ forms a negative cycle, with all other cycles in the matrix of length greater than one positive. Example 5 contains only negative cycles. In example 6, there is only one positive cycle, namely $a_{12}a_{23}a_{34}a_{45}a_{51}$, hence the GM conditions are satisfied.

Because we will be dealing extensively with index sets and with cycles in the rest of this paper, the following notation is introduced. We use the symbol J or I to denote the set of indices appearing in a cycle without regard to the order of such indices. $\vec{J}$ or $\vec{I}$ will be used to refer to the ordered set of indices in a cycle. Thus the cycle $a_{23}a_{31}a_{12}$ has an index set $J = (1, 2, 3)$ with ordered index set $\vec{J} = (2, 3, 1)$. The cycle $a_{13}a_{32}a_{21}$ then has an index set $I = (1, 2, 3)$ with ordered index set $\vec{I} = (1, 3, 2)$, so that $J = I$ but $\vec{J} \neq \vec{I}$. We will use interchangeably the terms "the cycle $\vec{I}$" and "the cycle $a_{i_1 i_2} a_{(i_2 \to i_1)}$" when $\vec{I} = (i_1, i_2, \cdots)$.

4. Because of the results derived by Metzler and Morishima, we know that for Metzlerian and Morishima matrices as special classes of GM matrices, stability implies Hicksian stability. This section is concerned with identifying other classes of GM matrices for which this proposition is true. An immediate result is the following.

Theorem 1. Let A be an indecomposable GM matrix satisfying either (i) all cycles in A of length greater than one are non-negative; or (ii) all cycles in A are non-positive. Then A stable implies that A is Hicksian.

Proof. The proof of (i) is well known (see [4]); in fact in this case A stable is equivalent to A Hicksian. To prove (ii) we use Maybee's determinantal formula. Negative cycles of length r enter into the expansion of principal minors of order r with sign $(-1)^r$, hence if A is GM with all cycles non-positive then every term in the expansion of an r^{th} order principal minor has sign $(-1)^r$ or 0. Further, A a GM matrix implies that diagonal elements in A are negative so that there exists a non-zero term in every principal minor in A, hence A is Hicksian.

So long as A contains no negative cycles of length greater than one or so long as A contains no positive cycles, under indecomposability the GM conditions guarantee that A stable implies that A is Hicksian. Theorem 2 below proves the proposition for a special case in which A contains both positive and negative cycles of length greater than one.

We first state several lemmas.

Lemma 1. Let A be an n x n GM matrix containing no positive cycles of length less than n. Then A stable implies that A is Hicksian.

Proof. By the Routh-Hurwitz conditions, A stable implies sign $|A| = (-1)^n$. Cycles of length n enter into the expansion of no principal minor of order less than n, and since all cycles of length less than n are non-positive, all terms in any principal minor of order i, $i = 1, \cdots, n-1$ have sign $(-1)^i$ or 0. By the GM conditions, diagonal elements in A are negative so that every i^{th} order principal minor of A contains a term in its expansion of sign $(-1)^i$, from which the lemma follows.

Lemma 2. Let A be an n x n indecomposable GM matrix. Let I be the index set of a negative cycle in A and let J be the index set of a positive cycle in A where $I \cap J \neq \emptyset$. Then if $\bar{I} = \bar{J}$, $J = S$ where $S = \{1, \cdots, n\}$, and all positive cycles in A are of length n.

Proof. Let $I = J = \{1, \cdots, s\}$ so that $\vec{I}$ and $\vec{J}$ are permutations of the first s integers, $s < n$. Without loss of generality reindex the negative cycle into $a_{12}a_{23} \cdots a_{s1}$. Since A is a GM matrix every positive cycle containing an index from I must contain all indices from I hence every positive cycle in the principal submatrix with index set I = J is of length s. A indecomposable means that given any $i \neq j$ $i, j = 1, \cdots, n$ there exists a non-zero chain $a(i \to j)$ in A. There exists $a_{ij} \neq 0$ for some $i \in J$, $j \in S \setminus J$. Then A indecomposable implies that there exists a non-zero cycle $a_{ij}a(j \to i)$ with index set K. This cycle cannot be negative since $K \cap J \neq \emptyset$ but $K \not\subseteq J$. If the cycle were negative, the GM conditions would be violated. Assume then that the cycle is positive. Since $I \cap K \neq \emptyset$ then $I \subseteq K$, i.e., the positive cycle contains all indices $1, \cdots, s$. Without loss of generality one can assume that this cycle contains the element a_{1s+1}, so we can write the positive cycle as $a_{1s+1}a(s+1 \to j^*)a(j^* \to 1)$ where j^* is the first index (following I) in the cycle which also appears in I. We first prove that $j^* = 2$. This is established as follows.

First, $j^* \neq 1$ since then the positive cycle could be written as $a_{1s+1}a(s+1 \to 1)$ with $1 \in K$ but $I \not\subseteq K$. If $j^* > 2$, then consider the cycle $a_{1s+1}a(s+1 \to j^*)a_{j^*j^*+1} \cdots a_{r1}$. This cycle does not contain the index 2. If this cycle is negative then it violates the GM conditions since it has indices in common with J but its index set is not contained in J. On the other hand, if the cycle is positive then again the GM conditions are violated since the cycle contains some but not all indices from I. It follows that $j^* = 2$, so that the positive cycle with index set K can be written as $a_{1,s+1}a(s+1 \to 2)a(2 \to 1)$.

Next write the positive cycle with index set J as $b(1 \to 2)b(2 \to 1)$. Thus a cycle is formed by $a_{1,s+1}a(s+1 \to 2)b(2 \to 1)$. Since s+1 appears in this cycle it cannot be negative by the GM condition. If the cycle is positive it must contain all indices from I by the GM conditions, which means that $b(2 \to 1)$ contains all indices from I. This means that a_{12} is an element both in the positive cycle with index set J and the negative cycle with index set I. Thus, sign $a_{23} \cdots a_{r1}$ is opposite to that of $b(2 \to 1)$. But now the cycle $a_{1,s+1}a(s+1 \to 2)\, a_{23} \cdots a_{r1}$ must be negative which violates the GM condition. It follows that $J = S$, and hence every positive cycle in A is of length n.

Theorem 2. Let A be an nxn indecomposable GM matrix. Let I be the index set of a negative cycle in A and let J be the index set of a positive cycle in A. If I = J then A stable implies that A is Hicksian.

Proof. From lemma 2, every positive cycle in A is of length n, which by lemma 1 establishes the theorem.

Certain other special cases of interest in economics follow from the preceding argument, including the case where A is "sign symmetric" (sign a_{ij} = sign a_{ji} $i \neq j$, where 0 is treated as a sign) and the case where A contains no zero entries.

Theorem 3. Let A be an $n \times n$ indecomposable GM matrix, satisfying sign symmetry. Then A is either a Metzler or a Morishima matrix, hence A is stable if and only if A is Hicksian.

Proof. Sign a_{ij} = sign a_{ji} $i \neq j$, hence every non-zero cycle in A of length two is positive. Since every index i $i \epsilon \{1, \cdots, n\}$ appears in a non-zero cycle, every index appears in some positive cycle of length two. It immediately follows that A can contain no negative cycles of length greater than one under the GM conditions. Indecomposability together with the condition all cycles of length greater than one are non-negative establishes the result.

Theorem 4. Let A be an $n \times n$ GM matrix satisfying $a_{ij} \neq 0$ for every $i, j = 1, \cdots, n$. Then A stable implies A is Hicksian.

Proof. If A contains no negative cycle of length greater than one then the result follows from theorem 1. Hence assume A contains a negative cycle of length greater than one. First assume A has no negative cycles of length greater than two and reindex a negative cycle into $a_{12}a_{21} < 0$. If $n = 2$ the theorem is immediate. If $n > 2$ then $a_{12}a_{23}a_{31} > 0$ and $a_{21}a_{32}a_{13} > 0$ which implies in turn either that $a_{23}a_{32} < 0$ and $a_{13}a_{31} > 0$ or $a_{23}a_{32} > 0$ and $a_{13}a_{31} < 0$. In either case the GM conditions are violated so that A must contain a negative cycle of length greater than two. Reindex such a cycle into $a_{12}a_{23} \cdots a_{r1} < 0$. By the GM conditions and $a_{ij} \neq 0$ $i, j = 1, \cdots, n$, $a_{12}a_{21} < 0$. If r is odd it follows that $a_{1r}a_{r,r-1} \cdots a_{21} > 0$ hence by theorem 2, A stable implies A is Hicksian. If r is even, consider $a_{12}a_{23} \cdots a_{r-2,r-1}a_{r-1,1}$. By the GM conditions this cycle must be negative. But r-1 is odd, hence $a_{1r-1}a_{r-1,r-2} \cdots a_{21}$ is positive and again theorem 2 applies, establishing the present theorem. (Note that theorem 4 is vacuous except for $n \leq 3$ or when A contains no negative cycles of length greater than one.)

Examples of GM matrices satisfying the conditions of theorems 2, 3 and 4 include the following.

$$\begin{bmatrix} - & - & 0 & 0 & + \\ + & - & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & - & - \\ - & 0 & 0 & + & - \end{bmatrix}$$

Theorem 2

$$\begin{bmatrix} - & + & - & 0 & 0 \\ + & - & 0 & 0 & 0 \\ - & 0 & - & + & 0 \\ 0 & 0 & + & - & + \\ 0 & 0 & 0 & + & - \end{bmatrix}$$

Theorem 3

$$\begin{bmatrix} - & - & + \\ + & - & - \\ - & + & - \end{bmatrix}$$

Theorems 2 and 4

We next characterize GM matrices containing both positive cycles and negative cycles of length greater than one. Of interest is the following basic result concerning matrices and cycles.

Lemma 3 [4]. Let A be an $n \times n$ indecomposable matrix containing both negative and positive cycles of length greater than one. Then there exists at least one index common to both a negative cycle of length greater than one and a positive cycle of length greater than one.

Under the GM conditions, this result can be sharpened considerably.

Lemma 4. Let A be an $n \times n$ indecomposable GM matrix. If A contains a positive cycle then every index i, $i \epsilon \{1, \cdots, n\}$ appears in a positive cycle in A.

Proof. Indecomposability requires that every index appears in some non-zero cycle of length greater than one. Suppose some index, say 1, appears only in a negative cycle. Denote the index set of the negative cycle by I_1. By indecomposability there exists $a_{ij} \neq 0$ $i \epsilon I_1, j \notin I_1$, with associated non-zero cycle $a_{ij}a(j \to i)$ and index set I_2. If this cycle is positive then by the GM conditions $I_1 \subseteq I_2$ hence the index 1 appears in a positive cycle. If the cycle is negative then again by indecomposability there exists a non-zero cycle with index set I_3 such that $(I_1 \cup I_2) \cap I_3 \neq \emptyset$. If this cycle is positive then either $I_1 \cap I_3 \neq \emptyset$ or $I_2 \cap I_3 \neq \emptyset$ implies by the GM conditions that either $I_1 \subseteq I_3$ or $I_2 \subseteq I_3$. But since $I_1 \cap I_2 \neq \emptyset$ by hypothesis, either case leads to the conclusion that $(I_1 \cup I_2) \subseteq I_3$ hence $1 \in I_3$. A continuation of this procedure until all indices in A are exhausted thus leads to the conclusion that A can contain no positive cycles if the index 1 appears only in a negative cycle, which leads to the desired contradiction.

Consider a GM matrix of the following type.

$$\begin{bmatrix} - & - & 0 & + & + & + \\ 0 & - & + & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & 0 \\ 0 & + & 0 & - & + & + \\ 0 & + & 0 & + & - & + \\ 0 & + & 0 & + & + & - \end{bmatrix}$$

One of the peculiarities of this matrix is that any positive cycle having an index in common with the negative cycle $a_{12}a_{23}a_{31}$ also contains the elements $a_{23}a_{31}$; e.g., $a_{14}a_{42}a_{23}a_{31}$, $a_{15}a_{56}a_{64}a_{42}a_{23}a_{31}$, etc. This property is in fact a distinguishing characteristic of indecomposable GM matrices, as indicated in lemma 5 below.

<u>Lemma 5</u>. Let A be an $n \times n$ indecomposable GM matrix. Let I be the index set of a negative cycle in A of length r, $r > 1$, and let J be the index set of a positive cycle in A of length s, $s < n$. If $I \cap J \neq \emptyset$ but $I \neq J$, then every positive cycle of length less than n in A with an index in common with I has r-1 <u>elements</u> in common with the negative cycle, each such positive cycle having the same elements in common.

Proof. Let $\vec{I}$ and $\vec{J}$ be the ordered index sets of negative and positive cycles such that $I \cap J \neq \emptyset$, $I \neq J$, where without loss of generality $\vec{I} = \{1, 2, \cdots, r\}$, $r > 1$, so that the negative cycle may be written as $a_{12}a_{23}\cdots a_{r1}$. Again without loss of generality let $a_{1,r+1}$ be an element in the positive cycle assumed to be of length $s<n$. We will show that under the conditions of the lemma, every positive cycle of length less than n with an index belonging to I contains the elements $a_{23}, \cdots, a_{r1}$. We begin by writing the positive cycle $\vec{J}$ as $a_{1,r+1}a(r+1 \to j^*)a(j^* \to 1)$ where j^* is the first index (following 1) in the cycle which also appears in I.

The idea of the proof is the following. In step 1, we show that $j^* = 2$. Step 1a shows that if $j^* > 2$ then for any $i \in J \backslash I$, there exists a negative cycle $a(i \to j)a(j \to i)$ containing some but not all indices in I. Step 1b shows that $j^* > 2$ implies that every positive cycle with index set contained in J has the index set J. Step 1c then establishes that $j^* > 2$ implies either that A is decomposable or $J = S$, where $S = \{1, \cdots, n\}$. Step 2 shows that the cycle $a_{1,r+1}a(r+1 \to 2)a_{23}\cdots a_{r1}$ is a positive cycle, where $a(r+1 \to 2)$ is the chain $a(r+1 \to j^*)$ appearing in $\vec{J}$. Step 3 then shows that any positive cycle of length less than n with an index in common with I contains the elements $a_{23}, \cdots, a_{r1}$.

<u>Step 1</u>. $j^* = 2$.

The positive cycle $\vec{J}$ is written as $a_{1,r+1}a(r+1 \to j^*)a(j^* \to 1)$ where j^* is the first index (following 1) in the cycle which also appears in I. Clearly $j^* \neq 1$ since then $I \cap J \neq \emptyset$ but $I \not\subseteq J$. Hence assume $j^* > 2$.

<u>1a</u>. $j^* > 2$ implies that for any $i \in J \backslash I$ there exists a negative cycle $a(i \to j)a(j \to i)$ containing some but not all indices in I.

If $j^* > 2$ then a non-zero cycle is formed by $a_{1,r+1}a(r+1 \to j^*)a_{j^*j^*+1}\cdots a_{r1}$. Since the index 2 does not appear in this cycle, by the GM condition the cycle must be negative. Consider next the product $a_{12}\cdots a_{j^*-1,j^*}a(j^* \to 1)$. This product is not a cycle since $a(j^* \to 1)$ must contain every index in I. Any product of this type which forms a closed loop can be factored into the product of cycles. (Note that all indices in the set $\{1, \cdots, j^*-1\}$ are repeated and that all other indices appearing in the product are distinct.) Write $a(j^* \to 1)$ as $a(j^* \to k_1^*)a(k_1^* \to k_2^*)\cdots a(k_{j^*-1}^* \to 1)$ where $k_1^*, k_2^*, \cdots, k_{j^*-1}^*$ are distinct indices from the set $\{1, 2, \cdots, j^*-1\}$. Each of these chains appears in a cycle within the product $a_{12}\cdots a_{j^*-1,j^*}a(j^* \to 1)$. Since $j^* > 2$ it follows that none of these cycles contains all indices from I, since in particular the index 1 appears only in cycles involving a_{12} and $a(k_{j^*-1}^* \to 1)$ and any such cycle does not contain all indices from I. Hence, by the GM conditions, each such cycle is negative. It thus follows that if $j^* > 2$ then given any index $i \in J\setminus I$, there exists a negative cycle $a(i\to j)a(j\to i)$, where each such cycle contains some but not all indices from I.

<u>1 b.</u> $j^* > 2$ implies that every positive cycle with index set K, $K \subseteq J$, satisfies $K = J$.

Assume that there is a positive cycle with index set K such that $K \subset J$, $K \neq J$. If there exists an index $i \in I \cap K$, then $K = J$ since every index in $J\setminus I$ appears in a negative cycle containing indices from I (and $I \subseteq K$ by the GM conditions). On the other hand, if K contains any indices from $J\setminus I$ then by the same argument $I \subset K$ so that $K = J$.

<u>1 c.</u> $j^* > 2$ implies that $J = S$ where $S = \{1, \cdots, n\}$ or A is decomposable.

Suppose J contains s indices where $s < n$. By indecomposability there exists $a_{ij} \neq 0$ $i \in J$, $j \in S\setminus J$ with associated non-zero cycle $a_{ij}a(j\to i)$. Write this cycle as $a_{ij}a(j\to k^*)\,a(k^*\to i)$ where k^* is the first index (following i) in the cycle which also appears in J. Any such cycle cannot be negative by the GM conditions since it includes indices from J, the index set of a positive cycle, as well as indices distinct from J. Further, if the cycle is positive, it must contain all indices from J since every index in J also appears in a negative cycle in the principal submatrix with index set J. If $i \in I$, $k^* \in I$, then $a_{ij}a(j\to k^*)a_{k^*,k^*+1}\cdots a_{i-1,i}$ forms a cycle which does not contain all indices from J, hence it violates the GM conditions. If $i \in I$, $k^* \in J\setminus I$, then since there exists a non-zero chain $a(k^*\to i)$ not containing all indices from I, again the cycle $a_{ij}a(j\to k^*)a(k^*\to i)$ does not include all indices from J and violates the GM conditions. Note finally that every index in $J\setminus I$ appears in a negative cycle, hence the two cases considered prove the assertion for $i \in J\setminus I$, $k^* \in J\setminus I$ and $i \in J\setminus I$, $k^* \in I$. Hence A is decomposable and the principal submatrix with index set J has no positive cycles of length less than s.

This completes the proof that under the conditions of the lemma, the positive cycle $\vec{J}$ written as $a_{1,r+1}a(r+1\to j^*)a(j^*\to 1)$ satisfies $j^* = 2$.

<u>Step 2</u>. $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1}$ is a positive cycle, where $a(r+1\to 2)$ is the chain $a(r+1\to j^*)$ appearing in the cycle $\vec{J}$.

The non-zero cycle $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1}$ contains all indices from I. Suppose that this cycle were negative. The positive cycle $\vec{J}$ is written as $a_{1,r+1}a(r+1\to 2)a(2\to 1)$. Then $a_{12}a(2\to 1)$ forms a cycle. Since $a_{12}a_{23}\cdots a_{r1} < 0$ and $a_{1,r+1}a(r+1\to 2)a(2\to 1) > 0$, while by hypothesis $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1} < 0$, it follows that $a_{12}a(2\to 1) > 0$.

But this contradicts the GM conditions since the index r+1 does not appear in the cycle $a_{12}a(2\to 1)$. Hence, $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1} > 0$.

<u>Step 3</u>. $j^* = 2$ implies that every positive cycle of length less than n with an index in common with I may be written as $a_{1k}a(k\to 2)a_{23}\cdots a_{r1}$, where $k \notin I$.

<u>3a</u>. $a_{1,r+1}a(r+1\to 2)a(2\to 1)$ has the property that the ordered index set of $a(2\to 1)$ is a permutation of the set $I = (1, \cdots, r)$.

$a_{1,r+1}a(r+1\to 2)a(2\to 1) > 0$ by hypothesis while $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1} > 0$ by step 2. Since $a_{12}a_{23}\cdots a_{r1} < 0$, this implies $a_{12}a(2\to 1) < 0$. Since $a_{12}a(2\to 1)$ includes all indices from I, its index set must be contained in the index set of $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1}$. But $a_{12}a(2\to 1)$ containing an index $k \notin I$ is consistent with its index set being contained in that of $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1}$ only if k is in the index set of $a(r+1\to 2)$. But this in turn implies that $a_{1,r+1}a(r+1\to 2)a(2\to 1)$ is not a cycle. Hence the ordered index set of $a(2\to 1)$ is a permutation of the set I.

Note that any positive cycle of length less than n having an index in common with I can be written as $a_{ij}a(j\to i+1)\,a(i+1\to i)$ $i \in I, j \notin I$, where $i+1$ is the first index (following i) in the cycle that appears in I.

(This holds if $i \in \{1, \cdots, r-1\}$. If $i = r$, then $i+1$ is replaced by the index 1.) It is also immediate from Step 2 and 3a that $a_{ij} a(j \to i+1) a_{i+1,i+2} \cdots a_{i-1,i} > 0$ and $a(i+1 \to i)$ has an ordered index set that is a permutation of I.

3b. In any positive cycle of length less than n, $a_{1k} a(k \to 2) a(2 \to 1)$ where $k \notin 1$, the ordered index set of $a(2 \to 1)$ is the set $(2, 3, \cdots, r, 1)$, i.e., $a(2 \to 1) = a_{23} a_{34} \cdots a_{r1}$.

From 3a, the ordered index set of $a(2 \to 1)$ is a permutation of the set I. To satisfy the GM conditions any such permutation must preserve the following properties of the principal submatrix with index set I.

(i) all non-zero chains $b(2 \to 1)$ in the principal submatrix of length less than r-1 have sign opposite to that of $a_{23} \cdots a_{r1}$; and

(ii) every non-zero cycle in the principal submatrix is negative.

Consider a permutation of I so that the ordered index set of $a(2 \to 1)$ is not equal to $(2, 3, \cdots, r, 1)$. Then there exists a non-zero element a_{ij} $i, j \in I$ such that $j > i+1$.

Then by (i) sign $(a_{23} \cdots a_{i-1,i} a_{ij} a_{j,j+1} \cdots a_{r1}) \neq$ sign $(a_{23} \cdots a_{r1})$ which implies sign $a_{ij} \neq$ sign $a_{i,i+1} \cdots a_{j-1,j}$. On the other hand, by (ii) $a_{ij} a_{j,j+1} \cdots a_{i-1,i} < 0$. But $a_{i,i+1} \cdots a_{j-1,j} a_{j,j+1} \cdots a_{i-1,i} < 0$. This contradiction establishes that $a(2 \to 1) = a_{23} \cdots a_{r1}$.

3c. Let K denote the index set of a positive cycle of length less than n, $a_{ij} a(j \to i+1) a(i+1 \to i)$ $i \in I$, $j \notin I$. Then $i \neq 1$ implies $K = J$ and every index in J appears in a negative cycle in the principal submatrix with index set J.

Form the product

$$a_{ij} a_K (j \to i+1) a_{i+1,i+2} \cdots a_{r1} a_{1,r+1} a_J (r+1 \to 2) a_{23} \cdots a_{i-1,i}$$

where $a_K(j \to i+1)$ is the chain $a(j \to i+1)$ from the ordered index set $\vec{K}$ and

$a_J(r+1 \to 2)$ is the chain $a(r+1 \to 2)$ from the ordered index set $\vec{J}$. This product is negative, since sign $a_{ij} a_K(j \to i+1) \neq$ sign $a_{i,i+1}$, and sign $a_{1,r+1} a(r+1 \to 2) \neq$ sign a_{12} while $a_{12} a_{23} \cdots a_{r1} < 0$. If no index in $a_K(j \to i+1)$ appears in $a_J(r+1 \to 2)$ then the product is a negative cycle containing some but not all indices from J thus violating the GM conditions. Hence there exists p_1 such that the product can be written as $a_{ij} a_K(j \to p_1) a_K(p_1 \to i+1) a_{i+1,i+2} \cdots a_{1,r+1} a_J(r+1 \to p_1) a_J(p_1 \to 2) a_{23} \cdots a_{i-1,i}$

The cycle $a_K(p_1 \to i+1) a_{i+1,i+2} \cdots a_{1,r+1} a_J(r+1 \to p_1)$ if positive violates the GM conditions. If negative, then all indices in $a_K(p_1 \to i+1)$ must appear in J and all indices $a_J(r+1 \to p_1)$ must appear in K. If the cycle is negative, then

$a_{ij} a_K(j \to p_1) a_J(p_1 \to 2) a_{23} \cdots a_{i-1,i} > 0$, hence this is not a cycle. This implies that the product can be written

$$a_{ij} a_K(j \to p_2) a_K(p_2 \to p_1) a_J(p_1 \to p_2) a_J(p_2 \to 2) a_{23} \cdots a_{i-1,i}.$$

Then $a_K(p_2 \to p_1) a_J(p_1 \to p_2) < 0$ since p_1 appears in a negative cycle, and the same argument can be applied so that every index in $a_K(p_2 \to p_1)$ appears in J, and every index in $a_J(p_2 \to p_1)$ appears in K. Clearly, a continuation of this procedure establishes the desired result.

3d. In the positive cycle $a_{ij} a(j \to i+1) a(i+1 \to i)$, $i = 1$.

If $i \neq 1$ then by 3c, all positive cycles with length less than n and an index in common with I have the index set J. Further, every index in J appears in a negative cycle of length greater than one in the principal submatrix with index set J.

Since A is indecomposable, there exists a non-zero cycle $a_{pq} a(q \to p)$ $p \in J$, $q \notin J$. This cycle must be a positive cycle of length n; if negative, it violates the GM conditions, while if of length less than n, it has no indices outside of J.

Given any element a_{uv} from the cycle $a_{1,r+1} a(r+1 \to 2) a(2 \to 1)$, form the product

$$a_{pg}a(q\to u)\,a_{uv}a(v\to p).$$

If this is not a cycle then there exists a cycle of length less than n involving a_{pq}. If it is a cycle, a_{uv} belongs to the cycle $a_{pq}a(q\to p)$. Since this holds for every element in $a_{1,r+1}a(r+1\to 2)\,a(2\to 1)$, $a_{pq}a(q\to p)$ cannot be a cycle of length n. Hence i = 1.

This completes the proof of the lemma.[4]

This lengthy and cumbersome proof, for which I apologize, not only provides a characterization of GM matrices, but also leads to several other cases where the GM conditions lead to the conclusion that stability of A implies that A is Hicksian. These cases again involve matrices that contain positive cycles as well as negative cycles of length greater than one.

Theorem 5. Let A be an n x n indecomposable GM matrix such that every index i, $i \in \{1,\cdots, n\}$, appears in a negative cycle in A of length greater than one. Then A stable implies that A is Hicksian.

Proof. If A contains no positive cycles then the theorem follows from Theorem 1. If A contains no positive cycles of length less than n, the result is immediate from lemma 1. Hence assume A contains a positive cycle of length less than n. In Step 1, we show that A does not contain two disjoint positive cycles.

Step 1. Reindex a negative cycle with an index in common with a positive cycle of length less than n into $a_{12}a_{23}\cdots a_{r1}$. By lemma 5, the positive cycle can be written as $a_{1,r+1}a(r+1\to 2)\,a_{23}\cdots a_{r1}$. Let the index set of this cycle be J_1.

Assume there exists a positive cycle with index set disjoint from J_1. Reindex a negative cycle into $a_{s+1,\,s+2}\cdots a_{t,s+1}$ and write the positive cycle with index set J_2, $J_2 \cap J_1 = \emptyset$, as $a_{s+1,\,t+1}a(t+1\to s+2)$ $a_{s+2,\,s+3}\cdots a_{t,s+1}$. By hypothesis, every index in $a(r+1\to 2)$ and in $a(t+1\to s+2)$ appears in a negative cycle in A of length greater than one. By indecomposability of A there exists $a_{ij}\neq 0$ $i\in J_1$, $j \in J_2$ with associated non-zero cycle $a_{ij}a(j\to i)$. Any such cycle must be positive since it contains an index in J_1 and an index distinct from J_1. Since every index in J_1 appears in a negative cycle of length greater than one, without loss of generality assume $i\in\{1,\cdots, r\}$. Then by lemma 5, $a_{ij}a(j\to i) = a_{1k}a(k\to 2)\,a_{23}\cdots a_{r1}$. Let p denote an index in $a(k\to 2)$ such that $p\in J_2$. Again without loss of generality assume $p\in\{s+1,\cdots, t\}$. Then by lemma 5, $a_{ij}a(j\to i) = a_{s+1,\ell}\,a(\ell\to s+2)\;a_{s+2,\,s+3}\cdots a_{t,\,s+1}$.

But this implies that $a_{1k}a(k\to 2)a_{23}\cdots a_{r1}$ contains within itself a cycle $a_{s+1,\ell}\,a(\ell\to s+1)$ hence $a_{ij}a(j\to i)$ is not a cycle. Hence, no positive cycles in A are disjoint.

Step 2. Assume next that A is stable but that A contains a principal minor of order i with sign $(-1)^{i+1}$. This principal minor must contain a positive cycle by Theorem 1. Then there exists within the principal minor a principal minor of order k, $k\leq i$, with sign $(-1)^{k+1}$, containing positive cycles only of length k. Denote the index set of this k - th order principal minor by K. Because no positive cycles in A are disjoint, the complementary principal minor in A of order n-k contains no positive cycles hence its sign is $(-1)^{n-k}$. In the expansion of $|A|$, we thus have the product of these two principal minors entering with sign $(-1)^{n+1}$ plus terms representing products of cycles with principal minors, the cycles being of the form $a_{ij}a(j\to i)$ $i\in K$, $j\notin K$. Each such cycle must be non-negative by the GM conditions, since every index in K appears in a positive cycle. No principal minor multiplying such a cycle can contain a positive cycle since positive cycles are not disjoint. Hence every term in the expansion of $|A|$ has sign $(-1)^{n+1}$ or 0. On the other hand, stability of A implies that sign $|A| = (-1)^n$. An identical argument holds for the case where A contains a principal minor which is zero, so that the theorem follows.

An example of a matrix satisfying the conditions of Theorem 5 is the following:

$$\begin{bmatrix} - & - & + & 0 \\ + & - & 0 & 0 \\ 0 & + & - & + \\ 0 & + & - & - \end{bmatrix}$$

Finally, we consider the case of combinatorially symmetric matrices. An n x n real matrix A is *combinatorially symmetric* if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$ $i, j = 1, \cdots, n$.

Theorem 6. Let A be an n x n indecomposable GM matrix which is combinatorially symmetric. Then A stable implies A is Hicksian.

Proof. We first show that if A contains both negative cycles of length greater than one and positive cycles, then no positive cycle of length less than n has an index in common with such a negative cycle.

Step 1. Assume that A contains a negative cycle of length greater than one with an index in common with a positive cycle of length less than n. Write the negative cycle as $a_{12}a_{23}\cdots a_{r1}$. Then by lemma 5, the positive cycle may be written as $a_{1,r+1}a(r+1\to 2)a_{23}\cdots a_{r1}$. By combinatorial symmetry, $a_{1r}a_{r,r-1}\cdots a_{21}$ forms a non-zero cycle which is negative by lemma 2. Further $a_{ij}a_{ji} < 0$ for each term a_{ij} in this cycle.

Similarly, by combinatorial symmetry, $a_{r+1,1} \neq 0$ and $a_{1,r+1}a_{r+1,1} < 0$, while $a_{ij}a_{ji} < 0$ for every element a_{ij} appearing in the chain $a(r+1\to 2)$.

Let $a(2\to r+1)$ represent the chain formed by the elements a_{ji} where a_{ij} is an element in $a(2\to r+1)$. Then

$a(2\to r+1)\,a_{r+1,1}a_{1r}a_{r,r-1}\cdots a_{32}$ forms a cycle. Note that $a_{1r}a_{r,r-1}\cdots a_{32}a_{21} < 0$ and $a_{1r}a_{r,r-1}\cdots a_{32}a_{12} > 0$.

Hence sign $a(2\to r+1)a_{r+1,1}\cdots a_{32}$ = sign $a(2\to r+1)a_{r+1,1}a_{12}$. Thus if $a(2\to r+1)a_{r+1,1}\cdots a_{32} > 0$, the GM conditions are violated by the cycle $a(2\to r+1)a_{r+1,1}a_{12} > 0$. If the cycle is negative, then A is decomposable by lemma 2. Hence no negative cycle in A of length greater than one has an index in common with a positive cycle of length less than n.

Step 2. If A contains a positive cycle of length n, and a negative cycle of length greater than one, then some element a_{ij} in the positive cycle has the index j in common with the negative cycle. Then by the GM conditions $a_{ij}a_{ji} < 0$ for any such element. But a_{ij} appearing in a negative cycle, a_{jk} also in the cycle of length n implies $a_{jk}a_{kj} < 0$, etc., so that $a_{rs}a_{sr} < 0$ for every element a_{rs} in the positive cycle. This implies in turn by Step 1 that A contains no positive cycles of length less than n, hence by Theorem 2, A stable implies A is Hicksian.

It has not yet been possible to prove the following conjecture.

Conjecture. Let A be an n x n indecomposable GM matrix. Then A stable implies that A is Hicksian.

Extensive work on this conjecture indicates that the key to establishing the conjecture might lie in matrices of type 4 depicted on page 6 above. For small dimension cases, the conjecture holds for such matrices, but the extension to n large has not been accomplished. Thus, for n = 3, the conjecture follows immediately. For n = 4, consider a matrix with sign pattern given by $\begin{bmatrix} - & - & + & + \\ + & - & 0 & 0 \\ 0 & + & - & + \\ 0 & + & + & - \end{bmatrix}$. Then $|A|$ may be written as $|A| = (a_{11}a_{22} - a_{12}a_{21})(a_{33}a_{44} - a_{34}a_{43}) - a_{21}a_{13}a_{34}a_{42} - a_{21}a_{14}a_{43}a_{32} + a_{44}(a_{21}a_{13}a_{32}) + a_{33}(a_{21}a_{14}a_{42})$. Since all non-zero cycles of length three and four are positive, the stability condition $|A| > 0$

implies that $a_{33}a_{44} - a_{34}a_{43} > 0$, hence all 2 x 2 principal minors are positive. This implies in turn that the only 3 x 3 principal minors which might be positive are those with indices (1, 2, 3) or indices (1, 2, 4). Let Δ_{123} denote the principal minor with indices 1, 2, 3. Then

$$|A| = a_{44}\Delta_{123} - a_{34}a_{43}(a_{11}a_{22} - a_{12}a_{21}) - a_{21}a_{13}a_{34}a_{42} - a_{21}a_{14}a_{43}a_{32} + a_{33}(a_{21}a_{14}a_{42}).$$

If $\Delta_{123} \geqq 0$, then every term in this expansion of $|A|$ is non-positive, which violates $|A| > 0$. A similar argument proves that $\Delta_{124} < 0$, hence A stable implies that A is Hicksian.

5. To indicate the relevance of the GM class for comparative statics analysis, assume that an economic model is specified in terms of variables $x_1, \cdots, x_n$, parameters $\alpha_1, \cdots, \alpha_n$ and functional relations $f^i(x_1, \cdots, x_n; \alpha_1, \cdots, \alpha_n)$ $i = 1, \cdots, n$. For given values $\alpha_1^o, \cdots, \alpha_n^o$ of the parameters, an equilibrium of the model is defined as a vector $(\bar{x}_1, \cdots, \bar{x}_n)$ such that

$$f_i(\bar{x}_1, \cdots, \bar{x}_n; \alpha_1^o, \cdots, \alpha_n^o) = 0 \qquad i = 1, \cdots, n.$$

Given a change in the j-th parameter, the resulting changes in the equilibrium values of the variables are obtained by solving the system

$$\sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k} \frac{d\bar{x}_k}{d\alpha_j} = - \frac{\partial f_i}{\partial \alpha_j} \qquad i = 1, \cdots, n.$$

Suppose that $\partial f_i / \partial \alpha_j = 0$ for $j \neq i$, so that each functional relation has associated with it its "own" parameter only. Then $d\bar{x}_i / d\alpha_i$ is of known sign for $i = 1, \cdots, n$ if and only if in the matrix $[\partial f_i / \partial x_k]$ the determinant of this matrix and all n - 1 x n - 1 principal minors of the matrix are of known sign.

In particular, assume that only the signs of the entries in $[\partial f_i / \partial x_k]$ are known (with diagonal elements negative). Then the postulate

of stability of this matrix implies through the "correspondence principle" that the signs $d\bar{x}_i / d\alpha_i$ are known for $i = 1, \cdots, n$ only if $[\partial f_i / \partial x_k]$ is a Hicksian matrix; further, for the special cases taken up in Section 4, it is known that the GM conditions are sufficient for signing $d\bar{x}_i / d\alpha_i$ $i = 1, \cdots, n$, under stability.

FOOTNOTES

1. A is quasi negative definite if $x'Ax<0$ for $x \neq 0$ (A not necessarily symmetric). A is D - stable if DA is a stable matrix for every diagonal matrix D with diagonal elements positive. A is totally stable if every principal submatrix of A is D - stable.

2. For completeness, we summarize Maybee's determinantal formula to show that negative cycles enter into the expansion of principal minors with "correct" sign.

   Let $S = \{1, \cdots, n\}$ and let $\alpha(r, S)$ denote the set of all strictly increasing multi-indices of length r in S. Thus, if $H \in \alpha(r,S)$, $H = \{h_1, h_2, \cdots, h_r\}$ where $1 \leq h_1 < h_2 \cdots < h_r \leq n$. Similarly, $\alpha(p, H)$ is the set of all strictly increasing multi-indices of length p in H. Let A_H denote the principal minor of A with index set H and let $A_{(H)}$ denote the sum of all cycles of length r in H. Then, given a fixed $K \in \alpha(n-1, S)$ the determinantal formula is given by

$$|A| = a_{k'k'} A_K + \sum_{r=0}^{n-1} (-1)^{n+1-r} \sum_{H \in \alpha(r,K)} A_H A_{(H')}$$

   where $a_{k'k'}$ is the diagonal element in A with index not contained in K and H' is the complement of H in S, and $A_{\emptyset} = 1$. Clearly negative cycles of length n enter into $|A|$ with sign $(-1)^n$. Applying the same formula to a principal minor of A of order r then leads to the conclusion that negative cycles of length r enter such principal minors with sign $(-1)^r$.

3. When A is decomposable, the cyclic characterization of a Morishima matrix does not necessarily correspond to the definition of a Morishima matrix given earlier. Thus $\begin{bmatrix} - & + & 0 \\ + & - & 0 \\ + & - & - \end{bmatrix}$ has all cycles of length greater than one

non-negative and yet is not a Morishima matrix.

4. To illustrate the role played by indecomposability in lemma 5, consider the following example:

$$\begin{bmatrix} - & - & 0 & + & 0 & + \\ 0 & - & - & 0 & 0 & + \\ - & 0 & - & 0 & + & + \\ - & 0 & 0 & - & + & + \\ 0 & + & 0 & - & - & + \\ 0 & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

   The matrix contains two positive cycles, $a_{14}a_{45}a_{52}a_{23}a_{31}$ and $a_{35}a_{54}a_{41}a_{12}a_{23}$. All other non-zero cycles are negative and the lemma fails because of decomposability.

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